THE GENERAL AND BASIC BOUNDARY VALUE PROBLEMS OF PLANE STEADY JET FLOWS AND THE CORRESPONDING NONLINEAR EQUATIONS

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The general boundary value problem, including known plane steady jet flows of an ideal incompressible fluid, is formulated. The simplest problem retaining all the specific features of the general problem, known as the basic problem, is separated from the general problem. The solution of the basic problem is reduced to solving a nonlinear integro-differential equation and also to solving nonlinear integral equations. Examples of flows whose determination is reduced to solving the basic problem are cited.

§1. THE GENERAL BOUNDARY VALUE PROBLEM. THE BASIC BOUNDARY VALUE PROBLEM

1.1. Determination of a jet flow in the parametric plane. The Zhukovskii function. Those flows whose boundaries consist of solid walls and free surfaces on which the pressures are constant are generally called jet flows. Here we shall consider a plane steady jet flow of an ideal fluid; it is also assumed that the fluid is incompressible and the flow region is simply connected. As is well known, in order to determine all the elements of such flows, it is sufficient to find the complex potential velocity of the flow $w = \varphi + i\psi$ at every point $z = x + iy$ of the flow region, that is, to find the relation $w = w(z)$. Finding the relation $w = w(z)$ is facilitated if we introduce the auxiliary complex variable $\zeta = \xi + i\eta$, which varies in some canonical region, together with an auxiliary function, and seek the relation $w = w(z)$ in the parametric form

$$w = w(\zeta), \qquad z = z(\zeta). \tag{1.1}$$

Therefore, the relation $w = w(z)$ will be taken here in the form (1.1). We shall take either the upper half-plane or the right quadrant of the upper half-plane as the canonical region of variation $\zeta = \xi + i\eta$. It is convenient to take the following function as the auxiliary function:

$$F(\zeta) = \frac{1}{\omega_0} \ln\left(\frac{1}{V_0}\frac{dw}{dz}\right) + F^\circ(\zeta). \tag{1.2}$$

Here ω_0 is some given number, V_0 the characteristic flow velocity, and $F^\circ(\zeta)$ the given function selected so that $F(\zeta)$ will be analytic in the canonical region. The function $F(\zeta)$ determined by the equality (1.2) will henceforth be called the Zhukovskii function. Since the flow is steady, the region of variation in the complex potential $w = \varphi + i\psi$ is always known and, consequently, we may consider the relation $w = w(z)$ known. The latter can be represented in the form

$$w = w(\zeta) = \varphi_0 f(\zeta). \tag{1.3}$$

Here φ_0 is a parameter having the dimension of the velocity potential which is characteristic of the given flow. Let the Zhukovskii function be known; then, making use of equalities (1.2) and (1.3), it is easy to find a function mapping the flow region onto the canonical region of the plane ζ:

$$z = z(\zeta) = \frac{\varphi_0}{V_0}\int_{\zeta_0}^{\zeta} e^{\omega_0(F^\circ(\zeta) - F(\zeta))} f'(\zeta)\, d\zeta. \tag{1.4}$$

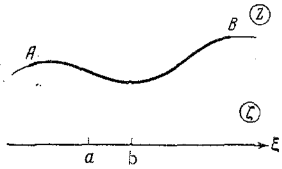

Fig. 1

For the complex velocity we have, according to (1.2), the expression

$$\frac{dw}{dz} = V_0 e^{\omega_0(F(\zeta) - F^\circ(\zeta))}. \tag{1.5}$$

Thus, in order to determine all the elements of the flow, it is sufficient to find the Zhukovskii function $F(\zeta)$ and the $w = w(\zeta)$.

1.2. The boundary conditions for the Zhukovskii function. In order to determine the form of the general boundary value problem for the Zhukovskii function, it is necessary to consider what conditions will be satisfied by the Zhukovskii function on different sections of the boundary of the canonical region, depending on the properties of the flow to which these sections correspond. Let AB be a part of the boundary of the flow region which corresponds in the plane $\zeta = \xi + i\eta$ to the segment $[a, b]$ of the ξ axis (Fig. 1). We shall denote the real part of $F(\zeta)$ by u and the imaginary part by v; hence

$$F(\zeta) = u + iv, \tag{1.6}$$

and we note, on the other hand, that

$$F(\zeta) = \frac{1}{\omega_0}\ln\left(\frac{V_s}{V_0} e^{-i\vartheta}\right) + F^\circ, \tag{1.7}$$

where V_s is the modulus of the velocity, ϑ the angle of inclination of the velocity to the real axis. Further, we shall denote the boundary values $F^\circ(\zeta)$ on the segment $[a, b]$ by $F^\circ(\xi)$. We note also that in the flow under consideration, the impermeable solid wall and the free surface are streamlines. We shall consider four cases depending on the form of the segment of the boundary AB.

1) The line AB is a solid impermeable wall. We set $\omega_0 = i$ and direct the axis so that the angle of inclination of the tangent to the curve AB will coincide with the angle of inclination of the velocity ϑ. We introduce the concept of relative curvature for the curve AB. Let $K^S(\vartheta)$ be the absolute value of the curvature of the line AB as a function of the angle of inclination of the tangent ϑ. Then the relative curvature of the curve AB will be the function

$$K(\vartheta) = K^s(\pi\sigma - \vartheta) / K^\circ. \tag{1.8}$$

Here K° is the curvature of the curve AB at some point of the line, and $\pi\sigma$ the angle of inclination of the tangent at this point. It is known that if S is the arc length of the curve AB, then the following equality is valid for $K^S(\vartheta)$:

$$d\vartheta/dS = \mp K_s^s(\vartheta). \tag{1.9}$$

Here we take the minus sign if ϑ decreases when S is increased, and the plus sign if ϑ increases. On the strength of the fact that AB is a streamline for the velocity and the velocity potential, the following relations hold:

$$V_s = \left|\frac{d\varphi}{dS}\right| \text{ on } AB; \qquad \varphi = \varphi_0 f(\xi), \qquad \xi \in [a, b]. \tag{1.10}$$

From relations (1.6)–(1.10), and introducing the dimensionless parameter

$$\lambda = \varphi_0 K^\circ / V_0 \tag{1.11}$$

we find that $F(\zeta)$ on the segment $[a, b]$ satisfies the boundary condition

$$\frac{du}{d\xi} \mp \lambda |f'(\xi)| e^{-\operatorname{Im} F^\circ(\xi)} K(u + \pi\sigma - \operatorname{Re} F^\circ(\xi)) e^v =$$
$$= \operatorname{Re} F^{\circ\prime}(\xi), \ \xi \in [a, b] \ (1.12). \tag{1.12}$$

2) The line AB is a free surface of a heavy fluid. Let us take $\omega_0 = 1$ and direct the coordinate axes so that the acceleration of gravity g will be directed toward the negative direction of the y axis and so that the angle of inclination of the tangent to the line AB will coincide with the angle of inclination of the velocity ϑ. Making use of the Bernoulli integral, on AB we have the equality

$$\tfrac{1}{2} V_s^2 + gy = C - p_0, \tag{1.13}$$

where p_0 is the pressure on the free surface and C is a constant. On the basis of obvious geometric considerations we find

$$dy/dS = \sin\vartheta. \tag{1.14}$$

We differentiate the equality (1.13) with respect to S and make use of the relations (1.6), (1.7), (1.10), and (1.14); then, introducing the dimensionless parameter

$$\lambda = g\varphi_0 / V_0^3 \tag{1.15}$$

we obtain for $F(\zeta)$ the boundary condition

$$\frac{du}{d\xi} - \lambda |f'(\xi)| e^{3\operatorname{Re} F^\circ(\xi)} \sin(v - \operatorname{Im} F^\circ(\xi)) e^{-3u} =$$
$$= \operatorname{Re} F^{\circ\prime}(\xi), \quad \xi \in [a, b]. \tag{1.16}$$

3) The line AB is a free surface subjected to capillary forces. We set $\omega_0 = -1$. Let p_1 be the pressure over the free surface and q the coefficient of surface tension; then, as is known [1],

$$p_s - p_0 = qK^s(\vartheta). \tag{1.17}$$

In this case the Bernoulli integral is of the form

$$p_1 - p_s = \tfrac{1}{2}\rho(V_0^2 - V_s^2) \tag{1.18}$$

From the relations (1.6), (1.7), (1.9), (1.10), (1.17), and (1.18), and introducing the dimensionless parameters

$$\lambda = \rho V_0 \varphi_0 / q, \qquad \mu = 2(p_1 - p_0) / \rho V_0^2 \tag{1.19}$$

we see that $F(\zeta)$ on the segment $[a, b]$ satisfies the boundary condition

$$\frac{du}{d\xi} \mp \lambda |f'(\xi)| [\operatorname{sh}(\operatorname{Im} F^\circ(\xi) - v) +$$
$$+ \frac{1}{2}\mu \exp(\operatorname{Im} F^\circ(\xi) - v)] = \operatorname{Re} F^{\circ\prime}(\xi),$$
$$\xi \in [a, b]. \tag{1.20}$$

4) The line AB is a solid permeable wall along which the velocity potential retains a constant value. In this case, it is obvious that

$$V_s = \left|\frac{d\psi}{dS}\right| \text{ on } AB; \quad \psi = -i\varphi_0 f(\xi), \quad \xi \in [a, b]. \tag{1.21}$$

We set $\omega_0 = 1$. Making use of the equality (1.21), also the relations (1.6)–(1.9), we find that on $[a, b]$ $F(\zeta)$ satisfies the boundary condition

$$\frac{du}{d\xi} \mp \lambda |f'(\xi)| e^{-\operatorname{Im} F^\circ(\xi)} K(u + \pi\sigma - \operatorname{Re} F^\circ(\xi)) e^v =$$
$$= \operatorname{Re} F^{\circ\prime}(\xi), \ \xi \in [a, b]. \tag{1.22}$$

where the parameter λ is determined by equality (1.12).

An analysis of the boundary conditions (1.13), (1.17), (1.21), and (1.23) makes it possible to establish the form of the general boundary problem for the Zhukovskii function including all stream flows whose boundaries consist of solid (generally speaking, curvilinear) walls and free surfaces which may be subject to gravitational or capillary forces. Indeed, if we take the upper half-plane as the canonical region of the variable $\zeta = \xi + i\eta$, then every solid wall or free surface of the plane of the flow will correspond to a segment on the real axis ξ (finite or infinite); on the other hand, the equalities (1.12), (1.16), (1.20), and (1.22) imply that the nonlinear part of the boundary problem for the function $F(\zeta) = u + iv$ on each such segment is the product of two functions, one of which depends only on u and the other only on v, the linear part contains the derivative in respect to ξ, or u, or v. If we take these two facts into consideration, also the possibility of arbitrary selection of the function $F^\circ(\zeta)$ in (1.2), then it is easy to verify that any boundary value problem for the Zhukovskii function, with the assumptions made above, will be a special case of the general boundary value problem.

1.3. The general nonlinear problem. Problem 1.1. The problem is to find the function $F(\zeta)$ which is analytic in the upper half-plane, bounded at infinity, and continuous on the real axis subject to the boundary conditions

$$\frac{1+\varkappa_\nu}{2}\frac{du}{d\xi}+\frac{1-\varkappa_\nu}{2}\frac{dv}{d\xi}=$$
$$=\lambda_\nu\gamma_\nu(\xi)\,U^\nu(u+\alpha_\nu(\xi))\,V^\nu(v+\beta_\nu(\xi))+\delta_\nu(\xi),\quad(1.23)$$
$$\xi\in\sigma_\nu\quad(\nu=1,2,\ldots,n),\quad\bigcup_{\nu=1}^{n}\sigma_\nu=(-\infty,\infty);\ \varkappa_\nu=\pm1.$$

where $\alpha_\nu(\xi)$, $\beta_\nu(\xi)$, $\gamma_\nu(\xi)$, $\delta_\nu(\xi)$, $U^\nu(u+\alpha)$ and $V^\nu(v+\beta)$ are given functions of their own arguments.

It is natural to call Problem 1.1 homogeneous if $F(\zeta)\equiv0$ is a solution of this problem. It is obvious that the following condition holds for the homogeneous problem:

$$\lambda_\nu\gamma_\nu(\xi)\,U^\nu(\alpha_\nu(\xi))\,V^\nu(\beta_\nu(\xi))+\delta_\nu(\xi)\equiv0$$
$$(\nu=1,2,\ldots,n).\qquad(1.24)$$

Henceforth, the parameters λ_ν will be denoted by λ. The coefficients of the problem may include some real parameters $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m$, that is, generally speaking,

$$\alpha_\nu(\xi)=\alpha_\nu(\xi,\varepsilon),\quad\beta_\nu(\xi)=\beta_\nu(\xi,\varepsilon),$$
$$\gamma_\nu(\xi)=\gamma_\nu(\xi,\varepsilon),\quad\delta_\nu(\xi)=\delta_\nu(\xi,\varepsilon),\qquad(1.25)$$

where ε denotes the set of parameters $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m$. In the general problem formulated the parameters λ and ε are assumed to be given.

1.4. The solution of the hydrodynamic problem. The boundary value problem with additional conditions imposed on the parameters. A solution of the boundary value problem (1.23) formulated above will not, generally speaking, yield a solution of the hydrodynamic problem to which it corresponds since the formulation of the majority of jet flow hydrodynamic problems is such that the parameters λ and ε are associated by some relationships with the unknown function. According to its origin, the latter may be geometrical or physical in nature. A part of these relations are represented by a transcendental equation of the form

$$\lambda_\nu=\tau_\nu/\chi_\nu(u,v,\varepsilon)\qquad(\nu=1,2,\ldots,n),\qquad(1.26)$$

whose number is usually equal to the number of parameters λ. Here τ_ν are given quantities and $\chi_\nu(u, v, \varepsilon)$ given functionals. In the general case, the remaining relations are implicit transcendental equations of the form

$$\chi^\nu(u,v,\varepsilon)=0\quad(\nu=1,2,\ldots,n),\qquad(1.27)$$

whose number is equal to the number of parameters ε. Examples of relations of the form (1.26), (1.27) will be given below when we consider jet flows with a curvilinear wall and jet flows of a heavy fluid [(2.11), (3.5), and (3.11)].

Thus, a solution of the jet flow hydrodynamic problem is equivalent to the following boundary value problem with conditions for the parameters λ and ε.

Problem 1.2. Find a solution of the general boundary value problem (1.23) under the assumption that the parameters λ and ε are associated with the unknown function by means of additional conditions (1.26), (1.27).

A natural way for solving Problem 1.2 is first to solve the general boundary value problem (1.23), after which the obtained solution $F(\zeta)=F(\zeta;\lambda,\varepsilon)=u(\xi,\eta;\lambda,\varepsilon)+iv(\xi,\eta;\lambda,\varepsilon)$ is substituted into relations (1.26), (1.27), and then the system of nonlinear transcendental equations for λ and ε obtained as a result of the substitution is solved.

1.5. The basic boundary value problem. Since solving the general boundary value problem involves difficulties, we shall isolate its simplest particular problem which at the same time retains the principal specific features of the general problem, both mathematical and physical.

We shall obtain such a problem if we set either $U^\nu(u+\alpha_\nu(\xi))\equiv0$ or $V^\nu(v+\beta_\nu(\xi))\equiv0$ in (1.23) on all segments σ_ν. Physically, this corresponds to the case in which all sections of the boundary corresponding to jet flow except one consist of solid rectilinear walls and free surfaces not acted upon by external forces. In this case, it is obvious that the function $F^\circ(\zeta)$ in equality (1.2) can always be expressed so that either $u=0$ or $v=0$ on all segments except one, and without loss of generality, we can take the segment $[-1, 1]$ as the latter. Thus, we approach the problem.

Problem 1.3. The problem is to find the function $F(\zeta)=u+iv$ which is analytic on the upper half-plane, bounded at infinity, and continuous on the real axis, with the boundary conditions

$$(1+\varkappa_\nu)v+(1-\varkappa_\nu)u=0,$$
$$\xi\in\sigma_\nu,\quad\varkappa_\nu=\pm1\quad(\nu=1,2,\ldots,n).\qquad(1.28)$$

$$\frac{du}{d\xi}=\lambda\gamma(\xi)\,U(u+\alpha(\xi))\,V(v+\beta(\xi))+\delta(\xi),$$
$$\xi\in[-1,1].\qquad(1.29)$$

where $\alpha(\xi)$, $\beta(\xi)$, $\gamma(\xi)$, $\delta(\xi)$, $U(u+\alpha)$, $V(v+\beta)$ are given functions of their own arguments and λ is the given parameter,

$$\bigcup_{\nu=1}^{n}\sigma_\nu\cup[-1,1]=(-\infty,\infty).$$

The boundary conditions of this basic problem contain four coefficients: $\alpha(\xi)$, $\beta(\xi)$, $\gamma(\xi)$, and $\delta(\xi)$ and one parameter (which is considered as being given). If the coefficients contain real parameters ε, that is, if

$$\alpha(\xi)=\alpha(\xi,\varepsilon),\ \beta(\xi)=\beta(\xi,\varepsilon),$$
$$\gamma(\xi)=\gamma(\xi,\varepsilon),\quad\delta(\xi)=\delta(\xi,\varepsilon),\qquad(1.30)$$

then they are considered given. The following condition holds for the homogeneous problem:

$$\lambda\,\gamma(\xi)U(\alpha(\xi))\,V(\beta(\xi))+\delta(\xi)\equiv0.\qquad(1.31)$$

When a solution of the given problem contains arbitrary constants, it is necessary to impose additional conditions on the desired function; their selection will depend on the physical peculiarities of the given flow. As one of such conditions, we can take the equality

$$u(\xi_0, 0) = 0, \qquad \xi_0 \in [-1, 1]. \tag{1.32}$$

In order for the basic problem to yield a solution of the corresponding hydrodynamic problem, it is necessary to satisfy the additional conditions (1.27) and one condition of the form

$$\lambda = \tau / \chi(u, v, \varepsilon) \tag{1.33}$$

where τ is a given parameter $\chi(u, v, \varepsilon)$ a given functional. Thus, a solution of problem (1.28), (1.29) is equivalent to solving the boundary value problem with additional conditions imposed on λ and ε.

Problem 1.4. The problem is to find a solution of the basic problem (1.28), (1.29) with the assumption that the parameters λ and ε are connected with the unknown function by the additional conditions (1.33), (1.27).

A natural way to solve problem (1.4) is first to solve the basic problem (1.28), (1.29); after this, the obtained solution $F(\zeta) = F(\zeta; \lambda, \varepsilon) = u(\xi, \eta; \lambda, \varepsilon) + iv(\xi, \eta; \lambda, \varepsilon)$ is substituted into the relationships (1.33), (1.27), and the system of equations obtained is solved for λ and ε.

§2. REDUCTION OF THE BASIC AND THE PRINCIPAL BOUNDARY VALUE PROBLEMS TO NONLINEAR INTEGRO-DIFFERENTIAL AND NONLINEAR INTEGRAL EQUATIONS

Let $u(\xi)$ be the real part of $F(\zeta)$ on the segment $[-1, 1]$ of the ξ axis, that is, we set

$$u(\xi) = u(\xi, 0), \qquad \xi \in [-1, 1]. \tag{2.1}$$

We shall consider that the function $u(\xi)$ is known; then determination of $F(\zeta)$ is reduced to solving the mixed problem or the Dirichlet problem for an analytic function. We shall assume that in the last case, the condition $F(\infty) = 0$ is imposed on $F(\zeta)$. If we make use of known solutions of these problems [2, 3], then it is not difficult to obtain an integral representation of $F(\zeta)$ through the function $u(\xi)$ in the upper half-plane. We introduce an integral of the form

$$\Omega(u \mid \zeta) = \frac{\omega(\zeta)}{\pi i} \int_{-1}^{1} \frac{u(t)}{t-\zeta} \frac{dt}{\omega(t)}, \qquad \zeta \,\overline{\in}\, [-1, 1] \tag{2.2}$$

where $\omega(\xi)$ is a real function and $\omega(\xi)$ a function which is analytic everywhere in the upper half-plane except, perhaps, at an infinitely distant point at which we assume there is a pole for the function and which satisfies the condition $\lim \omega(\zeta) = \omega(\xi)$ when $\zeta \to \xi$, $\xi \in [-1, 1]$. We emphasize by the notation $\Omega(u \mid \zeta)$ that this integral is not only a function of the point ζ, but also an operator with respect to $u(\xi)$. When it is necessary to stress only one of these factors or to note the dependence of this integral on $\omega(\zeta)$, then we shall also require the notation $\Omega(\zeta)$, Ωu or $\Omega(u, \omega \mid \zeta)$ for it.

In the special case, when $\omega(\zeta) \equiv 1$ or $\omega(\zeta) = -i\sqrt{\zeta^2 - 1}$, we shall denote this integral through

$$\Phi(u \mid \zeta) = \frac{1}{\pi i} \int_{-1}^{1} \frac{u(t)}{t-\zeta} dt, \qquad \zeta \,\overline{\in}\, [-1, 1].$$
$$\Psi(u \mid \zeta) = -\frac{\sqrt{\zeta^2-1}}{\pi} \int_{-1}^{1} \frac{u(t)}{t-\zeta} \frac{dt}{\sqrt{1-t^2}}, \qquad \zeta \,\overline{\in}\, [-1, 1]. \tag{2.3}$$

We shall understand $\sqrt{\zeta^2 - 1}$ to mean the branch which is analytic on the ξ axis of the plane ζ cut along the segment $[-1, 1]$ and such that $\sqrt{\zeta^2 - 1} = \zeta + O(\zeta^{-1})$ for large $|\zeta|$. Along with the integral $\Omega(u \mid \zeta)$, we introduce the singular integral

$$T(u \mid \xi) = \frac{\omega(\xi)}{\pi} \int_{-1}^{1} \frac{u(t)}{t-\xi} \frac{dt}{\omega(t)}, \qquad \xi \in [-1, 1] \tag{2.4}$$

which is understood in the sense of the Cauchy principal value. For this, we shall also make use of the notation $T(\xi)$, Tu, $T(u, \omega \mid \zeta)$, having the same sense as in the corresponding notation of the integral $\Omega(u \mid \zeta)$. In the two special cases for giving $\omega(\xi)$ noted previously, we shall denote the integral $T(u \mid \xi)$ through

$$J(u \mid \xi) = \frac{1}{\pi} \int_{-1}^{1} \frac{u(t)}{t-\xi} dt, \quad \xi \in [-1, 1].$$
$$I(u \mid \xi) = \frac{\sqrt{-\xi^2+1}}{\pi} \int_{-1}^{1} \frac{u(t)}{t-\xi} \frac{dt}{\sqrt{1-t^2}}, \quad \xi \in [-1, 1]. \tag{2.5}$$

If we make use of the introduced notation, then the above indicated integral representation $F(\zeta)$ through the function $u(\xi)$ can be represented in the general case (with the assumption made above) in the following form [2, 3]:

$$F(\zeta) = \Omega(u \mid \zeta); \tag{2.6}$$

however, if the order of the pole $\omega(\zeta)$ is equal to n, then $u(\xi)$ should satisfy the conditions

$$\int_{-1}^{1} u(t) t^{k-1} \frac{dt}{\omega(t)} = 0 \qquad (k = 1, 2, \ldots, m), \tag{2.7}$$

where $m = n - 1$ if $F(\zeta)$ is bounded on infinity and $m = n$ if $F(\infty) = 0$. If we pass to the limit in (1.6), (2.6) with $\zeta \to \xi$, $|\xi| \le 1$, then make use of the Sokhotskii formulas, then it is not difficult to verify that the real and imaginary parts of $F(\zeta)$ on the segment $[-1, 1]$ of the ξ axis are connected by the relations $v = -T(u \mid \xi)$. We now substitute the expression found in (1.29). Then, for determining $u(\xi)$ we obtain the nonlinear equation

$$u'(\xi) = \lambda\gamma(\xi) U(u(\xi) + \alpha(\xi)) V(-T(u \mid \xi) + \beta(\xi)) + \delta(\xi), \ \xi \in [-1, 1] \tag{2.8}$$

which is integro-differential and singular. According to (1.32), a solution of this equation must satisfy the condition

$$u(\xi_0) = 0, \quad \xi_0 \in [-1, 1] \tag{2.9}$$

and at times, conditions (2.7), if the solution of the basic boundary value problem is to satisfy these conditions. Henceforth, unless we state otherwise, we shall assume that the solution of Eq. (2.8) satisfies only condition (2.9). Thus, solving the basic boundary value problem is reduced to solving Eq. (2.8). Thus, it will be called the basic nonlinear integro-differential equation. The functions $\alpha(\xi)$, $\beta(\xi)$, $\gamma(\xi)$, and $\delta(\xi)$ are naturally called the coefficients of Eq. (2.8). It is natural to call Eq. (2.8) homogeneous if the coefficients satisfy the relation (1.31). In the general case, the coefficients depend on the parameters ε, that is, the relations (1.30) hold. The parameters λ and ε are considered given. Generally speaking, solving Eq. (2.8) will not be equivalent to solving the corresponding hydrodynamic problem. In order for a solution of Eq. (2.8) to be equivalent to the hydrodynamic problem, or, equivalently, the basic boundary value problem with additional conditions imposed on the parameters, it is essential that this solution satisfy the system of nonlinear functional equations obtained by substituting the functions $u(\xi, \eta; \lambda, \varepsilon)$, $v(\xi, \eta; \lambda, \varepsilon)$ of the equality (2.6), where $F(\zeta) = u(\xi, \eta; \lambda, \varepsilon) + iv(\xi, \eta; \lambda, \varepsilon)$, in the relationships (1.33), (1.27).

2.2. Transformation of Eq. (2.8) to nonlinear integral equations. In some cases, Eq. (2.8) can be reduced to a nonlinear integral equation. We shall consider three such cases.

1) In case the derivative $u'(\xi)$ is integrable, let the operator $T(u|\xi)$ be represented in the form

$$T(u\,|\,\xi) = T^*(u'\,|\,\xi) = -\int_{-1}^{1} u'(t)\,H(\omega\,|\,\xi,t)\,dt. \quad (2.10)$$

where $H(\omega|\xi, t)$ is a Fredholm type kernel, that is a function which is quadratically summable in the square $-1 \le \xi,\ t \le 1$. Moreover, let $M(u|\xi)$ be the operator inverse to $T(u|\xi)$. Then, if we introduce the function $v(\xi) = -T(u|\xi)$, it is easy to show that with the assumptions that have been made, solving Eq. (2.8) is reduced to solving the equation

$$v(\xi) = \lambda \int_{-1}^{1} \gamma(t)\,H(\omega\,|\,\xi,t)\,U(-M(v\,|\,t) + \alpha(t)) \cdot V(v(t) + \beta(t))\,dt + \delta^*(\xi),$$

$$\delta^*(\xi) = T^*(\delta\,|\,\xi), \qquad \xi \in [-1,1]. \quad (2.11)$$

2) Let the equality (2.10) and the following hold:

$$\alpha(\xi) = \alpha = \text{const}, \quad \delta(\xi) \equiv 0 .$$

In this case, if we introduce the function $Q(u)$ determined by the equalities

$$R(u) = \int_0^u \frac{du}{U(u+\alpha)}, \qquad Q(u) = \frac{dR^{-1}(u)}{du}$$

where $R^{-1}(u)$ is the operator inverse to $R(u)$, it is not difficult to verify that solving Eq. (2.8) is reduced to solving the equation

$$v(\xi) = \lambda \int_{-1}^{1} \gamma(t)\,H(\omega\,|\,\xi,t)\,V(v(t) + \beta(t))\,Q \cdot \left[\lambda \int_{\xi_0}^{t} \gamma(\tau)\,V(v(\tau) + \beta(\tau))\,d\tau\right] dt,$$

$$\xi \in [-1,1], \quad (2.12)$$

where $v(\xi)$ is the same function as in Eq. (2.11).

3) Let $U(u + \alpha)$ not depend on u, that is,

$$U(u(\xi) + \alpha(\xi)) = U(\alpha(\xi)) = U(\xi).$$

Then, Eq. (2.11) takes the form of a Hammerstein type equation:

$$v(\xi) = \lambda \int_{-1}^{1} \gamma^*(t)\,V(v(t) + \beta(t))\,H(\omega\,|\,\xi,t)\,dt + \delta^*(\xi),$$

$$\xi \in [-1,1], \quad \gamma^*(\xi) = \gamma(\xi)U(\xi). \quad (2.13)$$

Remark. If we make use of the known formulas for inverting a Cauchy type integral [2, 3], it is easy to establish that the following formula holds for the operator $M(v\,|\,\xi) = T^{-1}(v\,|\,\xi)$, included in Eq. (2.11):

$$M(v\,|\,\xi) = T(v, \omega^*\,|\,\xi) + C\omega^*(\xi), \qquad \omega^*(\xi) = \chi(\xi)\omega(\xi)$$

in which the value of the constant C and the form of the function $\chi(\xi)$ depend on the behavior of $\omega(\xi)$ and the unknown function $u(\xi)$ of Eq. (2.8) at the ends of the segment [−1, 1]. If the ratio $u(\xi)/\omega(\xi)$ has integrable singularities at both ends of the segment [−1, 1], then $\chi(\xi) = 1/\sqrt{1-\xi^2}$, and the constant C is determined from the condition (2.9). If the ratio $u(\xi)/\omega(\xi)$ is bounded on both ends of the segment [−1, 1], then $\chi(\xi) = \sqrt{1-\xi^2}$ and C = 0; however, the following condition should be satisfied:

$$\int_{-1}^{1} \frac{u(t)}{\omega(t)\sqrt{1-t^2}}\,dt = 0.$$

In case the ratio $u(\xi)/\omega(\xi)$ has an integrable singularity at the point $\xi = 1$ or $\xi = -1$, then, accordingly, $\chi(\xi) = \sqrt{1+\xi}/\sqrt{1-\xi}$ or $\chi(\xi) = \sqrt{1-\xi}/\sqrt{1+\xi}$ and C = 0. It is also not difficult to write an integral representation for the Zhukovskii function $F(\zeta)$ in terms of the function $v(\xi) = -T(u\,|\,\xi) = \operatorname{Im} F(\xi)$, $\xi \in [-1, 1]$.

2.3. Reducing the general problem to a system of nonlinear equations. Solving the general problem can be reduced to solving a system of nonlinear integro-differential equations. We shall show this through two examples.

As the first example, we can cite the problem of discontinuous flow around a finite number of curvilinear arcs in accordance with the Kirchhoff scheme. This problem was considered by L. I. Sedov and was reduced by him to a system of nonlinear integro-differential equations [1, 4].

As the second example, we shall consider cavitational flow of a heavy fluid along a plate in accordance with the Zhukovskii-Roshko scheme [1] with the assumption that the plate is oriented perpendicular to the velocity at infinity and that the walls which constrain the stream are parallel to the velocity at infinity. The flow under consideration is shown in Fig. 2, where the correspondence of points in the plane of the flow and the auxiliary plane is indicated. With this correspondence, the following formula for the complex potential will hold:

$$w = \varphi_0 (\zeta - \cos \varepsilon_0)^2,$$

$$\sqrt{\varphi_0} = \frac{1}{2} (\sqrt{\varphi_C} + \sqrt{\varphi_D}),$$

$$\cos \varepsilon_0 = \frac{\sqrt{\varphi_D} - \sqrt{\varphi_C}}{\sqrt{\varphi_D} + \sqrt{\varphi_C}}.$$

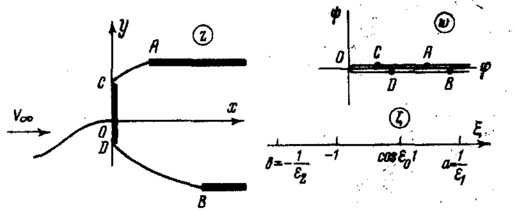

Fig. 2

We set $\omega_0 = 1$, $V_0 = V_\infty$ in equality (1.2) and choose the function $F^\circ(\zeta)$ so that it satisfies the conditions

$$\operatorname{Im} F^\circ(\xi) = \begin{cases} 0, & -\infty < \xi \leqslant -b,\ a \leqslant \xi < \infty, \\ -{}^1/_2\pi, & -1 \leqslant \xi < \cos \varepsilon_0, \\ {}^1/_2\pi, & \cos \varepsilon_0 < \xi \leqslant 1, \end{cases}$$

$$\operatorname{Re} F^\circ(\xi) = \begin{cases} \ln (V_\infty / V_2), & -b \leqslant \xi \leqslant -1, \\ \ln (V_\infty / V_1), & 1 \leqslant \xi \leqslant a, \end{cases}$$

where V_1 and V_2 are the velocities at the points C and D, respectively, then we introduce the functions determined by the equalities

$$u(\xi, 0) = u_1 (f_1 (\xi)),$$

$$f_1 (\xi) = (a + 1 - 2\xi) / (a - 1) \qquad 1 \leqslant \xi \leqslant a,$$

$$u(\xi, 0) = u_2 (f_2 (\xi)),$$

$$f_2 (\xi) = -(b + 1 + 2\xi) / (b - 1) \qquad -b \leqslant \xi \leqslant -1.$$

Then, the following integral representation will hold for the Zhukovskii function:

$$F(\zeta) = \frac{\omega(\zeta)}{\pi} \left[-\int_1^a \frac{u_1(f_1(t))}{t - \zeta} \frac{dt}{|\omega(t)|} + \int_{-b}^{-1} \frac{u_2(f_2(t))}{t - \zeta} \frac{dt}{|\omega(t)|} \right],$$

$$\omega(\zeta) = \sqrt{(\zeta^2 - 1)(\zeta - a)(\zeta + b)}.$$

Here we take that branch which takes real positive values on the segment (a, ∞) of the ξ-axis for $\omega(\zeta)$. We shall denote by $\omega_1(\zeta)$ and $\omega_2(\zeta)$ those branches of the functions

$$\omega_1 (\zeta) = i \sqrt{(\zeta^2 - 1)(\zeta - a_1)(\zeta - b_1)},$$

$$a_1 = \frac{a + 3}{a - 1}, \quad b_1 = \frac{2b + a + 1}{a - 1},$$

$$\omega_2 (\zeta) = -i \sqrt{(\zeta^2 - 1)(\zeta + a_2)(\zeta + b_2)},$$

$$b_2 = \frac{b + 3}{b - 1}, \quad a_2 = \frac{2a + b + 1}{b - 1},$$

which take purely imaginary positive or negative values on segments (b_1, ∞) or $(1, \infty)$, respectively, and by $\mu_1(\xi^*)$ and $\mu_2(\xi^*)$ the real functions

$$\mu_1 (\xi^*) = -\frac{a + b + 2 - (a - 1)\xi^*}{b - 1},$$

$$\mu_2 (\xi^*) = \frac{a + b + 2 - (b - 1)\xi^*}{a - 1}.$$

Making use of the notation introduced and boundary condition (1.16), it is not difficult to obtain the following system of equations for determining $u_1(\xi^*)$ and $u_2(\xi^*)$, where $\xi^* = f_1(\xi)$ if $1 \le \xi \le a$, and $\xi^* = f_2(\xi)$ if $-b \le \xi \le -1$:

$$u_1'(\xi^*) = \lambda_1 \gamma_1(\xi^*) e^{-3u_1(\xi^*)} \sin [T(u_1, \omega_1 \mid \xi^*) - i\Omega(u_2, \omega_2 \mid \mu_1) + \beta_1(\xi^*)], \quad (-1 \leqslant \xi^* \leqslant 1),$$

$$u_2'(\xi^*) = \lambda_2 \gamma_2(\xi^*) e^{-3u_2(\xi^*)} \sin [T(u_2, \omega_2 \mid \xi^*) - i\Omega(u_1, \omega_1 \mid \mu_2) + \beta_2(\xi^*)] \quad (-1 \leqslant \xi^* \leqslant 1),$$

$$\lambda_1 = \frac{g\varphi_0}{V_1^3}, \quad \gamma_1(\xi^*) = (a - 1)\left(\frac{a - 1}{2} \xi^* - \frac{a + 1}{2} + \cos \varepsilon_0\right),$$

$$\beta_1(\xi^*) = -\operatorname{Im} F^\circ [f_1^{-1}(\xi^*)],$$

$$\lambda_2 = \frac{g\varphi_0}{V_2^3}, \quad \gamma_2(\xi^*) = (b - 1)\left(\frac{b - 1}{2} \xi^* + \frac{b + 1}{2} + \cos \varepsilon_0\right),$$

$$\beta_2(\xi^*) = -\operatorname{Im} F^\circ [f_2^{-1}(\xi^*)].$$

The solution of system (2.14) must satisfy the condition $u_1(0) = 0$, $u_2(0) = 0$. As the basic given parameter, we shall take the inverse of the Froude number

$$\tau = gh / V_\infty^2$$

where h is the length of the plate. In order for the system (2.14) to provide a solution of the hydrodynamic problem under consideration with a given τ, it is essential that the following conditions be satisfied:

$$(F(\zeta) - F^\circ(\zeta)) = 0 \quad \text{for}$$

$$\lim_{\zeta \to \infty} \mid \exp [2 \lim_{\substack{\xi \to -1 \\ -1 \leqslant \xi < \cos \varepsilon_0}} (u(\xi, 0) - \operatorname{Re} F^\circ(\xi))] - \exp [2 \lim_{\substack{\xi \to 1 \\ \cos \varepsilon_0 < \xi \leqslant 1}} (u(\xi, 0) - \operatorname{Re} F^\circ(\xi))] = 2\tau,$$

$$2\lambda_1 i \int_{-1}^{1} (\xi - \cos \varepsilon_0) \exp (F^\circ(\xi) - u(\xi, 0))\, d\xi + \tau \exp [3 \lim_{\substack{\xi \to 1 \\ \cos \varepsilon_0 < \xi \leqslant 1}} (\operatorname{Re} F^\circ(\xi) - u(\xi, 0))] = 0,$$

$$2\lambda_2 i \int_{-1}^{1} (\xi - \cos \varepsilon_0) \exp (F^\circ(\xi) - u(\xi, 0))\, d\xi + \tau \exp [3 \lim_{\substack{\xi \to -1 \\ -1 \leqslant \xi < \cos \varepsilon_0}} (\operatorname{Re} F^\circ(\xi) - u(\xi, 0))] = 0.$$

Conditions (2.15) yield five real transcendental equations for determining the parameters λ_1, λ_2, ε_0, $\varepsilon_1 = 1 / a$ and $\varepsilon_2 = 1 / b$ $(0 \leqslant \varepsilon_1, \varepsilon_2 \leqslant 1)$.

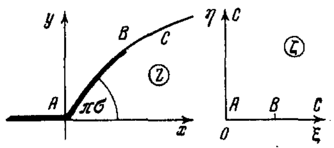

Fig. 3

§3. **Examples of jet flows whose determination is reduced to solving the basic boundary value problem.** In the general case, determining a jet flow is reduced, as can be seen from the foregoing, to solving the general boundary value problem; however, determining some whole classes of jet flows is reduced to solving the basic boundary value problem. Here we give three classes of jet flows.

3.1. Jet flows of a weightless fluid with a curvilinear wall. Let us consider a flow whose boundaries consist of a finite number of n rectilinear walls, one curvilinear wall, and a free stream coinciding with it; there are no stagnation points on the curvilinear wall. We shall direct the coordinate axis as shown in Fig. 3, which shows a part of the boundary of the flow consisting of the curvilinear wall AB, also the rectilinear segment and the free stream BC adjacent to it. We shall determine the relative curvature $K(\vartheta)$ of the arc AB from the equality (1.8) where $\pi\sigma$ and $K°$ are the angle of inclination of the tangent and the curvature of the wall at point A, respectively. We shall assume further that $K(\vartheta)$ is an even function, which does not restrict generality, since it can always be extended from a region of positive values of the argument to a region of negative values (or vice versa) in an even manner. We shall take as the canonical region of variation in ζ the right quadrant of the upper half-plane in accordance with the points shown in Fig. 3 in which all rectilinear boundaries of the flow are transferred to the imaginary η axis so that the segment σ_ν ($\nu = 1, 2, \ldots, n$) of the η axis corresponds to each rectilinear section of the flow boundary. The angle of inclination of the velocity on a rectilinear segment of the flow velocity corresponding to the segment σ_ν will be denoted by ϑ_ν ($\nu = 1, 2, \ldots, n$). We set $\omega_0 = i$ in equality (1.2) and take

$$F°(\zeta) = \pi\sigma + 2i\sigma \ln \frac{\zeta}{1 - i\sqrt{\zeta^2 - 1}} + F_0(\zeta),$$

where the function $F_0(\zeta)$ satisfies the conditions

$$\operatorname{Re} F_0(i\eta) = \vartheta_\nu, \quad \eta \in \sigma_\nu \quad (\nu = 1, 2, \ldots, n);$$
$$\operatorname{Re} F_0(\xi) = 0, \quad \xi \in [-1, 1],$$
$$\operatorname{Im} F_0(\xi) = 0, \quad \xi \in [1, \infty].$$

Now, we also introduce the function

$$\gamma(\xi) = f'(\xi)\left(\frac{1 + \sqrt{1 - \xi^2}}{\xi}\right)^{2\sigma} e^{-\operatorname{Im} F_0(\xi)}, \quad \xi \in [0, 1]$$

then extend it to segment [−1, 0] evenly and denote the function constructed in this way, varying on the segment [−1, 1], again by $\gamma(\xi)$.

We now extend the function $F(\zeta)$ which is analytic on the entire surface of the half-plane, considering that it should satisfy condition (1.12); then we obtain for determining $F(\zeta)$ the boundary value problem

$$v = 0, \quad |\xi| \in [1, \infty];$$
$$\frac{du}{d\xi} = \lambda\gamma(\xi) K(u) e^v, \quad \xi \in [-1, 1], \tag{3.1}$$

where the parameter λ is given by the equality (1.11). The problem obtained here is a special case of the basic problem when $\varkappa = 1$, $U(u + \alpha) = K(u)$, $V(v + \beta) = e^v$. In this case, the integral representation (2.6) is of the form $F(\zeta) = \Psi(u \mid \zeta)$, $\xi \in [-1, 1]$ for $F(\zeta)$; from this $v = -I(u \mid \xi)$, $\xi \in [-1, 1]$, and, consequently, solving the problem obtained here is reduced to solving the equation

$$u'(\xi) = \lambda\gamma(\xi) K(u(\xi)) e^{-I(u \mid \xi)}, \quad \xi \in [-1, 1] \tag{3.2}$$

under the condition $u(0) = 0$, an odd solution being sought. We shall take the following quantities as the given geometric parameters:

$$\tau = K°S_0, \quad \tau_\nu = L_\nu / S_0 \quad (\nu = 1, \ldots, n). \tag{3.3}$$

where L_ν are the lengths of rectilinear segments which correspond in the ζ plane to the segments σ_ν on the η axis. As a physical parameter, we take the quantity

$$\varkappa_0 = 2(p_\infty - p_0)/\rho V_\infty^2 = V_0^2/V_\infty^2 - 1, \tag{3.4}$$

which is generally called the cavitation number; here p_∞, p_0, V_∞, V_0 are the pressure and velocity at infinity and at the free stream, respectively. We shall make use of η_ν and $\eta_{\nu+1}$ to denote the ends of the segments σ_ν ($\nu = 1, 2, \ldots, n$) and η_0 a point on the η axis which corresponds in the plane of the flow to an infinitely remote point. We introduce the functions

$$\gamma_\nu(\eta) = \eta^{-2\sigma}(1 + \sqrt{1 + \eta^2})^{2\sigma} f'(i\eta) e^{-\operatorname{Im} F_0(i\eta)},$$
$$\eta \in \sigma_\nu \quad (\nu = 1, \ldots, n).$$

Making use of the notation introduced here, it is not difficult to obtain the relations

$$\tau = \lambda\chi(v, \varepsilon), \qquad \chi(v, \varepsilon) \equiv \int_0^1 |\gamma(\xi)| e^{v(\xi, 0)} d\xi,$$
$$\chi(v, \varepsilon) - \tau_\nu \chi^\nu(v, \varepsilon) = 0,$$
$$\chi^\nu(v, \varepsilon) \equiv \int_{\eta_\nu}^{\eta_{\nu+1}} |\gamma_\nu(\eta)| e^{v(0, \eta)} d\eta \qquad (\nu = 1, \ldots, n), \tag{3.5}$$
$$\varkappa_0 = \eta_0^{-4\sigma}(1 + \sqrt{1 + \eta_0^2})^{4\sigma} \exp(-2\operatorname{Im} F_0(i\eta_0) + 2v(0, \eta_0)),$$

in which the functions $\gamma(\xi)$, $\gamma_\nu(\xi)$, and the quantities η_ν ($\nu = 1, 2, \ldots, n$) depend, in general, on the parameters ε. Relations (3.5) are additional conditions imposed on the parameters λ and ε which must be satisfied by a solution of problem (3.1) if it is a solution of the hydrodynamic problem with given parameters (3.3) and (3.4). We shall write the integral equation for $v(\xi) = -I(u \mid \xi)$. In this case we note that when $u(\xi)$ is an even function which is bounded and does not vanish at the ends of the segment [−1, 1] and $u[0] = 0$, the operator which is the inverse of $I(u|\xi)$ will be $I^{-1}(u \mid \xi) = -J(u \mid \xi)$ (refer to the comment in regard to 2.2 in §2). We also not that it is not difficult to obtain the following formula for $I(u|\xi)$ by integrating by parts:

$$I(u \mid \xi) = -\int_{-1}^{1} u'(t) H(\xi, t)\, dt,$$

$$H(\xi, t) = \frac{1}{\pi} \ln \frac{|t - \xi|}{1 - \xi t + \sqrt{(1 - \xi^2)(1 - t^2)}}. \tag{3.6}$$

We substitute $u'(t)$ from Eq. (3.2) into (3.6) and introduce the function $v(\xi) = -I(u \mid \xi)$. We then obtain the integral equation

$$v(\xi) = \lambda \int_{-1}^{1} \gamma(t) H(\xi, t) K(J(v \mid t)) e^{v(t)} dt,$$

which is a special case of Eq. (2.11).

If we take into consideration that $\operatorname{Re} F(\xi) = v(\xi, 0) = v(\xi)$, $\xi \in [-1, 1]$, and make use of the Schwarz formula, then it is easy to obtain the integral representation $F(\zeta) = \Phi(v \mid \zeta)$ for the function $F(\zeta)$.

3.2. Jet flows of a heavy fluid with rectilinear boundaries. Let us consider a flow of a heavy fluid whose boundaries consist of a finite number n of rectilinear solid walls and one free surface. We shall direct the y axis vertically upward. As the canonical region of the variable ζ, we shall take the upper half-plane with a correspondence of points such that the free surface will correspond with the segment [−1, 1] on the ξ axis so that the remaining part of the real axis is divided into segments σ_ν ($\nu = 1, 2, \ldots, n$), each of them corresponding in the plane of the flow to a rectilinear section of the boundary with the angle of inclination of the velocity along it ϑ_ν ($\nu = 1, 2, \ldots, n$). We shall set $\omega_0 = 1$ in (1.2) and choose the function $F°(\zeta)$ so that it will satisfy the conditions

$$\operatorname{Im} F°(\xi) = \vartheta_\nu, \quad \xi \in \sigma_\nu \quad (\nu = 1, \ldots, n);$$
$$\operatorname{Re} F°(\xi) = 0, \xi \in [-1, 1]. \tag{3.7}$$

Then, taking condition (1.16) into account and introducing the function

$$\gamma(\xi) = f'(\xi), \quad \beta(\xi) = -\operatorname{Im} F°(\xi), \quad \xi \in [-1, 1],$$

we obtain the following boundary value problem for determining $F(\zeta)$:

$$v = 0, \ |\xi| \in [1, \infty];$$

$$\frac{du}{d\xi} = \lambda\gamma(\xi) \sin(v + \beta(\xi)) e^{-3u}, \quad \xi \in [-1, 1]. \tag{3.8}$$

Here the parameter λ is given by the equality (1.15). The problem obtained here is a special case of the basic problem when $\varkappa = 1$,

$U(u+\alpha) = e^{-3u}$, $V(v+\beta) = \sin(v+\beta)$. The representation $F(\zeta) = \Psi(u \mid \zeta)$, holds for the function $F(\zeta)$ in this case and, consequently, solving problem (3.7) is reduced to solving the equation

$$u'(\xi) = \lambda\gamma(\xi)\sin(-I(u \mid \xi) + \beta(\xi))\, e^{-3u(\xi)}, \quad \xi \in [-1, 1] \quad (3.9)$$

with the condition (2.9). We shall take as the given parameters

$$\tau = gL_0 / V_\infty, \qquad \tau_\nu = L_\nu / L_0 \qquad (\nu = 1, \ldots, n) \quad (3.10)$$

where L_0 is the characteristic dimension (the value of the abscissa or the ordinate of one of the points A or B of the free surface will be taken as this dimension), V_∞ is the velocity at infinity, L_ν are the lengths of the rectilinear segments which correspond in the ζ plane with the segments $\sigma_\nu = [\xi_\nu, \xi_{\nu+1}]$ on the ξ axis ($\nu = 1, 2, \ldots, n$). We assume that the velocity at infinity is parallel to the axis of abscissas and that the point $\zeta = \infty$ corresponds to the point $z = \infty$. We introduce the function

$$\gamma_\nu(\xi) = f'(\xi)\, e^{\operatorname{Re} F^\circ(\xi)}, \qquad \xi \in \sigma_\nu \qquad (\nu = 1, \ldots, n).$$

Making use of the notation introduced, it is not difficult to obtain the relations

$$\tau\chi_0(u, \varepsilon) - \lambda\chi(u, v, \varepsilon) = 0; \quad \chi_0(u, \varepsilon) = \exp(F(\infty) - F^\circ(\infty)),$$

$$\tau_\nu\chi(u, v, \varepsilon) - \chi^\nu(u, \varepsilon) = 0,$$

$$\chi^\nu(u, \varepsilon) \equiv \int_{\xi_\nu}^{\xi_{\nu+1}} |\gamma_\nu(\xi)|\, e^{-u(\xi, 0)}\, d\xi \qquad (\nu = 1, \ldots, n), \quad (3.11)$$

$$\chi(u, v, \varepsilon) \equiv \pm \int_{-1}^{1} \gamma(\xi)\, e^{-u(\xi, 0)} \sin\left(\frac{\pi\sigma}{2} - \beta(\xi) - v(\xi, 0)\right) d\xi,$$

in which $\sigma = 0$ or $\sigma = 1$, depending on whether L_0 is the length of an ordinate or an abscissa, and the sign in front of the integral is chosen so that the condition $\chi(u, v, \varepsilon) > 0$ is satisfied. Relations (3.11) are additional conditions imposed on λ and ε which must be satisfied by the solution of the problem (3.8) if it is a solution of the hydrodynamic problem with the given parameters (3.10).

Let the unknown function $u(\xi)$ be bounded and not vanish at the ends of the segment $[-1, 1]$. Then the operator inverse to $I(u|\xi)$ with conditions (2.9) is of the form $I^{-1}(u \mid \xi) = -J(u \mid \xi) + J(u \mid \xi_0)$ (refer to the remark in ¶2.2). Taking this into consideration, also (3.6), we obtain the integral equation for the function $v(\xi) = -I(u \mid \xi)$

$$v(\xi) = \lambda \int_{-1}^{1} \gamma(t) H(\xi, t) \sin(v(t) + \beta(t))\, e^{-3(J(v|t) - J(v|\xi_0))}\, dt,$$

$$\xi \in [-1, 1],$$

where $H(\xi, t)$ is a function included in equality (3.6). The integral representation expressing the Zhukovskii function by means of the function $v(\xi)$ is in this case of the form $F(\zeta) = i\Phi(v \mid \zeta) - J(v \mid \xi_0)$.

We multiply Eq. (3.9) by $e^{-3u(\xi)}$, then we take the integral from sides of the obtained equality, selecting the constant of integration from condition (2.9). Then, after differentiating with respect to ξ, we have

$$u'(\xi) = \frac{\lambda\gamma(\xi)\sin(-I(u \mid \xi) + \beta(\xi))}{1 + 3\lambda \int_{\xi_0}^{\xi} \gamma(\tau)\sin(-I(u \mid \tau) + \beta(\tau))\, d\tau}.$$

We substitute the expression for $u'(\xi)$ found here in equality (3.6) and introduce the function $v(\xi) = -I(u \mid \xi)$. As a result, we obtain the integral equation

$$v(\xi) = \lambda \int_{-1}^{1} \frac{\gamma(t) H(\xi, t) \sin(v(t) + \beta(t))}{1 + 3\lambda \int_{\xi_0}^{t} \gamma(\tau)\sin(v(\tau) + \beta(\tau))\, d\tau}\, dt, \quad \xi \in [-1, 1].$$

This equation is a special case of Eq. (2.12).

3.3. Jet flows with rectilinear boundaries in the presence of capillary forces. Let us consider a flow of weightless fluid whose boundaries consist of a finite number n of rectilinear solid walls and one free surface subject to capillary forces. We choose the canonical region and the correspondence of points just as in the preceding flow. We set $\omega_0 = -i$ in equality (1.2) and make the function $F^\circ(\zeta)$ subject to the conditions

$$\operatorname{Re} F^\circ(\xi) = -\vartheta_\nu, \quad \xi \in \sigma_\nu \ (\nu = 1, \ldots, n);$$

$$\operatorname{Im} F^\circ(\xi) = 0, \quad \xi \in [-1, 1].$$

Here ϑ_ν and σ_ν are the same as in (3.7). We introduce the functions

$$\gamma(\xi) = f'(\xi), \qquad \delta(\xi) = \operatorname{Re} F^{\circ\prime}(\xi), \quad \xi \in [-1, 1]$$

and the parameters λ and μ determined by equalities (1.19). Making use of the notation introduced and taking condition (1.20) into consideration, we obtain the following boundary value problem for determining $F(\zeta)$:

$$u = 0, \quad |\xi| \in [1, \infty];$$

$$\frac{du}{d\xi} = -\lambda\gamma(\xi)\left(\operatorname{sh} v - \frac{1}{2}\mu e^{-v}\right) + \delta(\xi), \quad \xi \in [-1, 1]. \quad (3.12)$$

The problem obtained here is a special case of the basic problem, when $\varkappa = -1$; $U(u+\alpha) \equiv 1$, $V(v+\beta) = -\operatorname{sh} v + {}^1/_2\mu e^{-v}$. In this case, the integral representation $F(\zeta) = \Phi(u \mid \zeta)$, holds for the function $F(\zeta)$, from which $v = -J(u \mid \xi)$, $\xi \in [-1, 1]$, and, consequently, solving problem (3.12) is reduced to solving the equation

$$u'(\xi) = \lambda\gamma(\xi)(\operatorname{sh} J(u \mid \xi) + {}^1/_2\mu e^{J(u|\xi)}) + \delta(\xi), \quad \xi \in [-1, 1] \quad (3.13)$$

with the condition (2.9). The additional conditions for parameters λ and ε for problem (3.12) will be of the same type as conditions (3.5) and (3.11) for problems (3.1) and (3.8). We assume that the unknown function $u(\xi)$ of Eq. (3.13) vanishes at the ends of the segment $[-1, 1]$. In this case, after integrating by parts, we have the following formula for $J(u|\xi)$:

$$J(u \mid \xi) = -\int_{-1}^{1} u'(t) \ln|\xi - t|\, dt. \quad (3.14)$$

We substitute $u'(t)$ from Eq. (3.13) into (3.6) and introduce the function $v(\xi) = -J(u|\xi)$. We then obtain the integral equation

$$v(\xi) = \lambda \int_{-1}^{1} \gamma(t)\left(\frac{1}{2}\mu e^{-v(t)} - \operatorname{sh} v(t)\right) \ln|\xi - t|\, dt + \delta^*(\xi),$$

$$\xi \in [-1, 1]$$

$$\delta^*(\xi) = -\int_{-1}^{1} \delta(t) \ln|\xi - t|\, dt.$$

The integral representation expressing the Zhukovskii function in terms of the function $v(\xi)$ is in this case of the form $F(\zeta) = i\Psi(v \mid \zeta)$.

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